

Monotone meshfree methods for linear elliptic equations in non-divergence form via nonlocal relaxation

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Outline

- 1 Introduction
 - Background
- 2 Basic Ideas
 - Meshfree Finite Difference Method
 - Analytical Results
- 3 Numerical Results
 - 2d Examples
 - 3d Examples
- 4 Degenerate Case Examples
 - Boundary Degeneration
 - Internal Degeneration
- 5 Conclusion

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Main Goal

Solve the second-order linear elliptic equation in non-divergence form

$$\begin{cases} -Lu(\mathbf{x}) := -\sum_{i,j=1}^d a^{ij}(\mathbf{x})\partial_{ij}u(\mathbf{x}) = f(\mathbf{x}) & \mathbf{x} \in \Omega \\ u(\mathbf{x}) = g(\mathbf{x}) & \mathbf{x} \in \partial\Omega \end{cases},$$

for an open bounded domain $\Omega \subset \mathbb{R}^d$. The matrix $A(\mathbf{x}) = (a^{ij}(\mathbf{x}))_{i,j=1}^d$ is assumed to be symmetric and positive definite satisfying the uniform ellipticity condition

$$\lambda|\boldsymbol{\xi}|^2 \leq \boldsymbol{\xi}^T A(\mathbf{x})\boldsymbol{\xi} \leq \Lambda|\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} \in \mathbb{R}^d, \forall \mathbf{x} \in \Omega,$$

for positive constants λ, Λ with ratio $\varrho := \lambda/\Lambda \leq 1$.

Denote $M(\mathbf{x}) := (A(\mathbf{x}))^{1/2}$.

Applications

Non-divergence form elliptic equations

- Arise in probability and stochastic processes¹
- Are linearizations of Hamilton-Jacobi-Bellman (HJB) equations (with applications in the fields of optimal control and finance)²
- Are linearizations of the Monge-Ampère equation (with applications to the optimal transportation problem and geometry)³

¹[Cabré, 2008, Fleming and Soner, 2006]

²[Fleming and Soner, 2006]

³[Caffarelli and Gutiérrez, 1997]

Main Challenges

- 1 In general, the equation cannot be recast into a divergence form

$$-\operatorname{div}(\tilde{A}(\mathbf{x})\nabla u) + \mathbf{b}(\mathbf{x}) \cdot \nabla u,$$

e.g. when $A(\mathbf{x})$ is discontinuous.

We cannot easily apply FEM to solve the equation in non-divergence form.

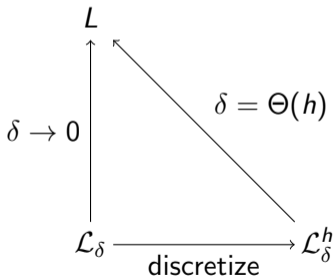
- 2 When $\varrho \ll 1$, the equation is near degenerate. Maximum principle preserving finite difference method in general **requires wide stencils**⁴.

⁴[Kocan, 1995] gave an estimate of the stencil width for the existence of positive-type finite difference method, and it grows ϱ^{-1} in 2d (similarly in a recent work [Mirebeau, 2016]) and $\varrho^{-5/2}$ in 3d.

Our Work

Main Idea:

Nonlocal Relaxation
+
Robust Discretization



Main Results:

- 1 Monotone meshfree method:** has guaranteed convergence.
- 2 Significant improvement** in theoretical analysis of the existence of positive stencils ($\varrho^{-1/2}$ stencil width growth for both 2d and 3d).
- 3 Practical:** sparse linear system, successfully implemented in 2d and 3d for a variety of geometries and coefficient matrices $A(\mathbf{x})$, applicable to the **near-degenerate regime** $\varrho \ll 1$; new experiments on degenerate elliptic equations.

Nonlocal Laplace operator

When $A(\mathbf{x}) = I_{d \times d}$, we get the Laplace operator:

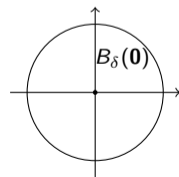
$$\Delta u(\mathbf{x}) = \sum_{i=1}^d \partial_{ii} u(\mathbf{x}).$$

The nonlocal Laplace operator⁵ is given by

$$\tilde{\mathcal{L}}_{\delta} u(\mathbf{x}) = \int_{B_{\delta}(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|\mathbf{y}|}{\delta}\right) (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y},$$

where γ is a nonnegative kernel with

$$\int_{B_1(\mathbf{0})} |\mathbf{y}|^2 \gamma(|\mathbf{y}|) d\mathbf{y} = 2d.$$



⁵[Du et al., 2012, Silling, 2000]

Nonlocal Laplace operator

$$\tilde{\mathcal{L}}_{\delta}u(\mathbf{x}) = \int_{B_{\delta}(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|\mathbf{y}|}{\delta}\right) (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y},$$

It can be shown that

$$\tilde{\mathcal{L}}_{\delta}u(\mathbf{x}) \rightarrow \Delta u(\mathbf{x}) \quad \text{as } \delta \rightarrow 0.$$

Nonlocal Relaxation to Elliptic Equations

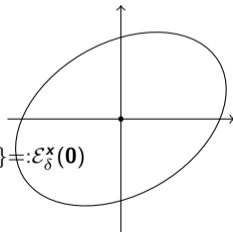
For general $A(\mathbf{x})$, the nonlocal elliptic operator⁶ can be defined as

$$\begin{aligned} \mathcal{L}_\delta u(\mathbf{x}) &= \int_{B_\delta(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|\mathbf{z}|}{\delta}\right) (u(\mathbf{x} + M(\mathbf{x})\mathbf{z}) - u(\mathbf{x})) d\mathbf{z} \\ &= \int_{\mathcal{E}_\delta^{\mathbf{x}}(\mathbf{0})} \frac{1}{\delta^{d+2}} \gamma\left(\frac{|M(\mathbf{x})^{-1}\mathbf{y}|}{\delta}\right) \det(M(\mathbf{x}))^{-1} (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y} \\ &:= \int_{\mathcal{E}_\delta^{\mathbf{x}}(\mathbf{0})} \rho_\delta(\mathbf{x}, \mathbf{y}) (u(\mathbf{x} + \mathbf{y}) - u(\mathbf{x})) d\mathbf{y}. \end{aligned}$$

It can be shown that

$$\mathcal{L}_\delta u(\mathbf{x}) \rightarrow Lu(\mathbf{x}) \quad \text{as } \delta \rightarrow 0.$$

$$\{\mathbf{y} \in \mathbb{R}^d : M(\mathbf{x})^{-1}\mathbf{y} \in B_\delta(\mathbf{0})\} =: \mathcal{E}_\delta^{\mathbf{x}}(\mathbf{0})$$



⁶[Nochetto and Zhang, 2018]

Minimal Positive Stencil

$$\begin{cases} \text{minimal} & \implies \text{sparsity} \\ \text{positive} & \implies \text{stability} \end{cases}$$

We usually use **truncation error** to analyze the consistency of a method. Then if the method also satisfies the **discrete maximum principle**, according to the Lax equivalence theorem, we will have a convergent method.

Minimal Positive Stencil

Minimal stencils are beneficial for the sparsity of the linear system matrix, resulting in a lower memory consumption and a faster solution of a linear system.

In formula

$$\mathcal{L}_\delta^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{N}(\mathbf{x}_i)} \beta_{j,i} (u(\mathbf{x}_j) - u(\mathbf{x}_i))$$

where $\mathcal{N}(\mathbf{x}_i)$ is some neighborhood of \mathbf{x}_i ,
we need a small $\#\{j : \beta_{j,i} \neq 0\}$.

Minimal Positive Stencil

We aim to obtain a **positive** stencil, because a positive stencil automatically satisfies the discrete maximum principle.

In formula

$$\mathcal{L}_\delta^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{N}(\mathbf{x}_i)} \beta_{j,i} \left(u(\mathbf{x}_j) - u(\mathbf{x}_i) \right)$$

we need $\beta_{j,i} \geq 0$ for all j .

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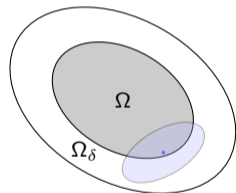
Notations

Let Ω_δ be the extended domain.

Point cloud $X = \{\mathbf{x}_1, \dots, \mathbf{x}_M\} \subset \Omega_\delta$ be given. Meshfree just means that no information about connection of points is provided.

Point cloud contains two types of points:

- 1 Interior points $\{\mathbf{x}_1, \dots, \mathbf{x}_N\}$ (in Ω),
- 2 Boundary points $\{\mathbf{x}_{N+1}, \dots, \mathbf{x}_M\}$ (in $\Omega_\delta \setminus \Omega$).

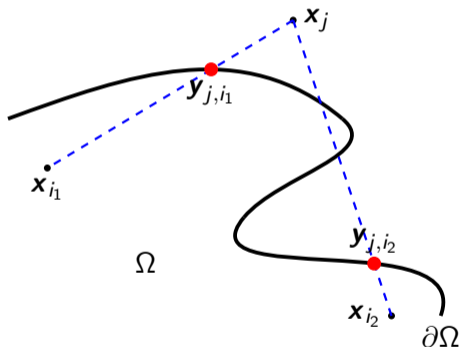


For each interior point, two steps are needed:

- 1 Define which points are its neighbors (vary for $A(\mathbf{x})$),
- 2 Select a stencil (using a minimization problem).

Boundary Treatment

We consider each boundary point \mathbf{x}_j around \mathbf{x}_i as the closest projection $\mathbf{y}_{j,i}$ from \mathbf{x}_j to \mathbf{x}_i at the boundary.



Then apply the boundary condition to these mapped boundary points to proceed.

Moving Least Squares

Moving least squares method⁷ finds a unique stencil via a quadratic minimization problem, take the approximation of Laplace operator as an example:

$$\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \mathcal{S}_{\delta,h,p}(\mathbf{x}_i)} \sum_j \frac{\beta_{j,i}^2}{W_\delta(|\mathbf{x}_j - \mathbf{x}_i|)},$$

where

$$\mathcal{S}_{\delta,h,p}(\mathbf{x}_i) := \left\{ \{\beta_{j,i}\} : \tilde{\mathcal{L}}_\delta^h u(\mathbf{x}_i) = \tilde{\mathcal{L}}_\delta u(\mathbf{x}_i) \quad \forall u \in \mathcal{P}_p(\mathbb{R}^d) \right\},$$

$$W_\delta(|\mathbf{x}_j - \mathbf{x}_i|) = \frac{1}{\delta^{d+2}} \gamma \left(\frac{|\mathbf{x}_j - \mathbf{x}_i|}{\delta} \right),$$

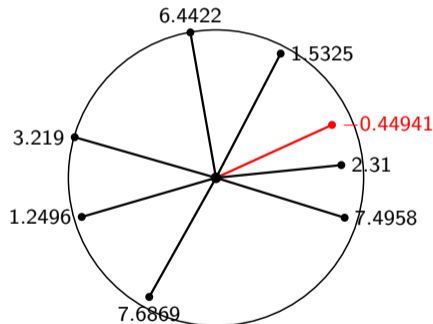
$$\tilde{\mathcal{L}}_\delta^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{B}_\delta(\mathbf{x}_i)} \beta_{j,i} (u(\mathbf{x}_j) - u(\mathbf{x}_i)); \quad \tilde{\mathcal{L}}_\delta u(\mathbf{x}) \rightarrow \Delta u(\mathbf{x}) \quad \text{as } \delta \rightarrow 0.$$

⁷[Liu et al., 1996, Mirzaei et al., 2012, Trask et al., 2019]

Moving Least Squares

$$\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \mathcal{S}_{\delta,h,p}(\mathbf{x}_i)} \sum_j \frac{\beta_{j,i}^2}{W_{\delta}(|\mathbf{x}_j - \mathbf{x}_i|)},$$

- 1 In general, least squares approaches do not select minimal stencils.
- 2 When ϱ is small, the number of points in the neighborhood grows very large, therefore quadratic minimization results in a much denser system.
- 3 In general, least squares approaches do not select positive stencils.



ℓ_1 Type Minimization Problem

We use the following minimization problem⁸ to select a unique stencil for interior point \mathbf{x}_i and p the order of the polynomial space:

$$\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \mathcal{S}_{\delta,h,p}(\mathbf{x}_i)} \sum_j \frac{\beta_{j,i}}{\rho_{\delta}(\mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_i)},$$

where

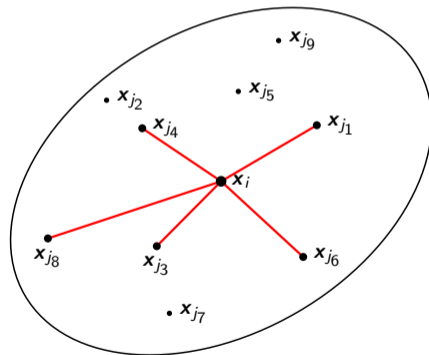
$$\mathcal{S}_{\delta,h,p}(\mathbf{x}_i) := \left\{ \{\beta_{j,i}\} : \beta_{j,i} \geq 0 \text{ and } \mathcal{L}_{\delta}^h u(\mathbf{x}_i) = \mathcal{L}_{\delta} u(\mathbf{x}_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d) \right\},$$

$$\mathcal{L}_{\delta}^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{E}_{\delta}^{\mathbf{x}_i}(\mathbf{x}_i)} \beta_{j,i} (u(\mathbf{x}_j) - u(\mathbf{x}_i)).$$

⁸[Seibold, 2008, Davydov and Schaback, 2018]

ℓ_1 Type Minimization Problem (cont.)

$$\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \mathcal{S}_{\delta,h,\rho}(\mathbf{x}_i)} \sum_j \frac{\beta_{j,i}}{\rho_{\delta}(\mathbf{x}_i, \mathbf{x}_j - \mathbf{x}_i)},$$



ℓ_1 Type Minimization Problem (cont.)

$$\mathbf{y}_{j,i} = \begin{cases} \mathbf{x}_j & , \mathbf{x}_j \in \bar{\Omega} \\ \text{projection from } \mathbf{x}_j \text{ to } \mathbf{x}_i \text{ at } \partial\Omega & , \mathbf{x}_j \in \Omega_\delta \setminus \bar{\Omega} \end{cases}$$

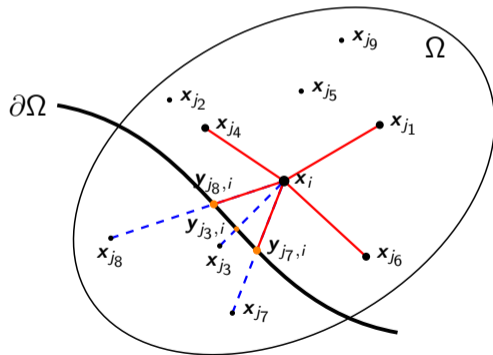
and

$$\bar{S}_{\delta,h,p}(\mathbf{x}_i) := \left\{ \{\beta_{j,i}\} : \beta_{j,i} \geq 0 \text{ and } \mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) = \mathcal{L}_\delta u(\mathbf{x}_i) \forall u \in \mathcal{P}_p(\mathbb{R}^d) \right\},$$

$$\mathcal{L}_{\delta,\Omega}^h u(\mathbf{x}_i) = \sum_{\mathbf{x}_j \in \mathcal{E}_\delta^{\mathbf{x}_i}(\mathbf{x}_i)} \beta_{j,i} (u(\mathbf{y}_{j,i}) - u(\mathbf{x}_i)).$$

ℓ_1 Type Minimization Problem (cont.)

Then the minimization problem becomes $\{\beta_{j,i}\} = \arg \min_{\{\beta_{j,i}\} \in \overline{S}_{\delta,h,p}(\mathbf{x}_i)} \sum_j \frac{\beta_{j,i}}{\rho_\delta(\mathbf{x}_i, \mathbf{y}_{j,i} - \mathbf{x}_i)}$,



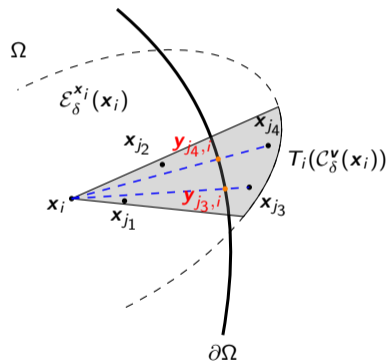
Existence of Solution Condition

Theorem 1 (Ye-Tian, 2023)

In $d = 2$ or $d = 3$, there exists a positive constant c (which only depends on d)⁹ such that if

$$\delta \geq ch\rho^{-1/2},$$

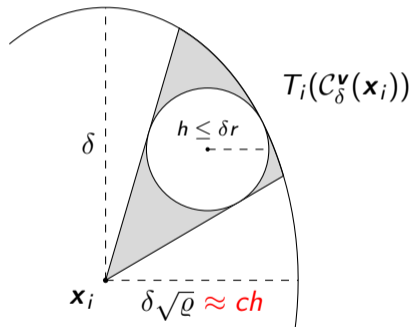
then $S_{\delta,h,2}(\mathbf{x}_i)$ and $\bar{S}_{\delta,h,2}(\mathbf{x}_i)$ are not empty.



⁹In our manuscript, $c = 3.614$ in 2d and $c = 4.450$ in 3d.

Significance/Implication of the Previous Theorem

- 1 For a fixed ϱ , $\delta = \Theta(h)$, which is a **robust discretization**.
- 2 When $\varrho \rightarrow 0$, the stencil width is $\varrho^{-1/2}$. The size of the elliptic region is **near-optimal**.¹⁰



¹⁰This theorem **significantly improves** the results in [Kocan, 1995] and [Mirebeau, 2016].

Convergence Estimate

Theorem 2 (Ye-Tian, 2023)

In $d = 2$ or $d = 3$, assume $\bar{S}_{\delta,h,p}(\mathbf{x}_i)$ is not empty, let u be the real solution and u_δ^h be the solution solved by the discrete operator and $C > 0$ is a generic constant.

1 If $p \geq 2$ and $u \in C^{2,\alpha}(\bar{\Omega})$ for $\alpha \in (0, 1]$, then

$$\max_{\mathbf{x}_i \in \Omega} |u(\mathbf{x}_i) - u_\delta^h(\mathbf{x}_i)| \leq C |u|_{C^{2,\alpha}(\bar{\Omega})} (\sqrt{\varrho})^{-\alpha} h^\alpha.$$

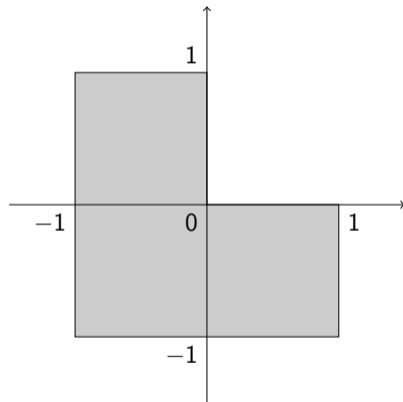
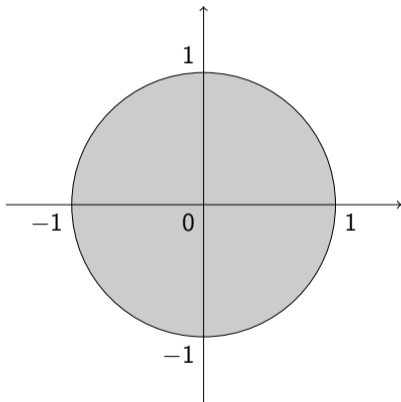
2 If $p \geq 3$ and $u \in C^{3,\alpha}(\bar{\Omega})$ for $\alpha \in (0, 1]$, then

$$\max_{\mathbf{x}_i \in \Omega} |u(\mathbf{x}_i) - u_\delta^h(\mathbf{x}_i)| \leq C |u|_{C^{3,\alpha}(\bar{\Omega})} (\sqrt{\varrho})^{-(1+\alpha)} h^{1+\alpha}.$$

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Domains

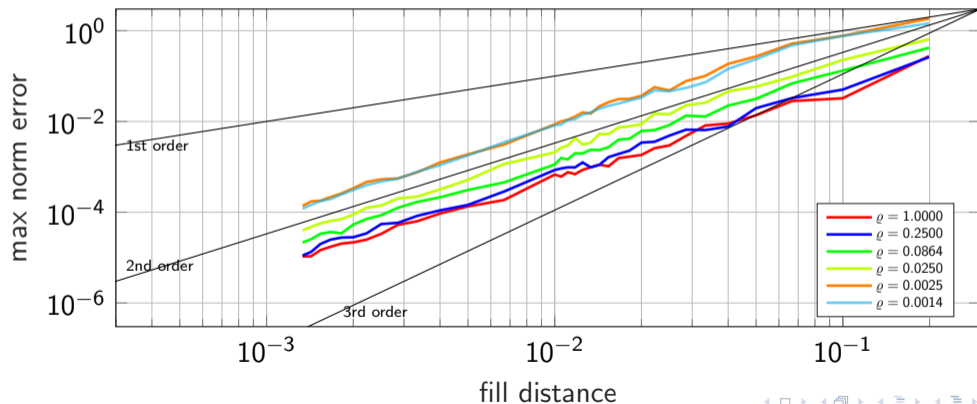


Continuous Coefficient Matrices

#	$A(\mathbf{x})$	$\varrho = \lambda/\Lambda$ for $\mathbf{x} \in [-1,1]^2$
0	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1.0000
1	$\begin{pmatrix} 1 - 0.5 x_1 & 0 \\ 0 & 0.25 + 0.25 x_2 \end{pmatrix}$	0.2500
2	$\frac{1}{2.21} \begin{pmatrix} 2 - x_1 & 0.5 \\ 0.5 & 0.5 + 0.5 x_2 \end{pmatrix}$	0.0864
3	$\begin{pmatrix} 1 - 0.5 x_1 & 0 \\ 0 & 0.025 + 0.025 x_2 \end{pmatrix}$	0.0250
4	$\begin{pmatrix} 1 - 0.5 x_1 & 0 \\ 0 & 0.0025 + 0.0025 x_2 \end{pmatrix}$	0.0025
5	$\frac{1}{2.001} \begin{pmatrix} 2 - x_1(0.5 - x_2) & 0.025 \\ 0.025 & 0.01 + 0.0025 x_1 \exp(x_2) \end{pmatrix}$	0.0014

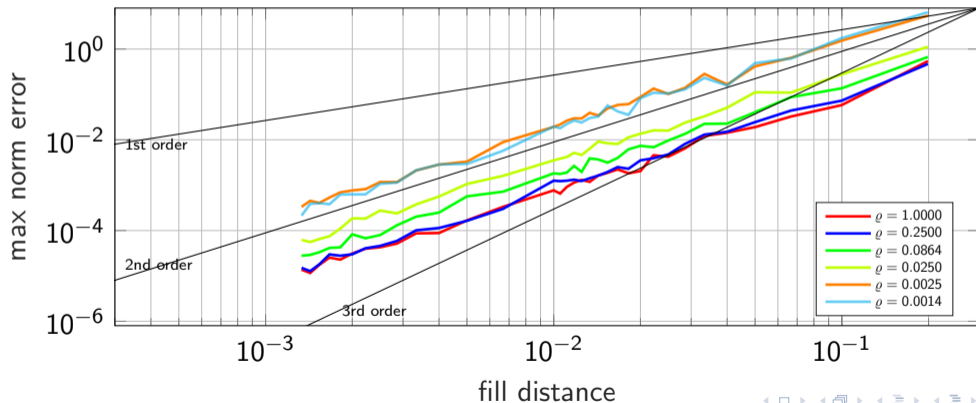
Error Graph for Continuous Coefficient Matrices

Unit Disk Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$



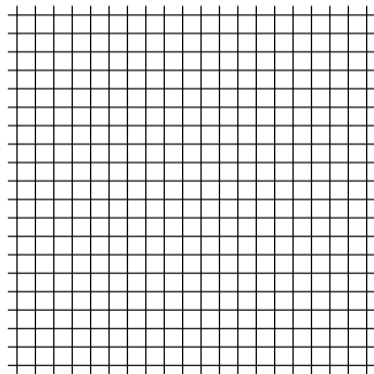
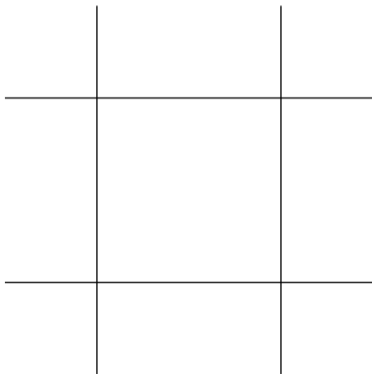
Error Graph for Continuous Coefficient Matrices

L-shaped Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$



Discontinuous Coefficient Matrices

Divide the domain into blocks and define piecewise constant coefficient matrices with respect to the blocks.



Discontinuous Coefficient Matrices

We tested for the following $A(\mathbf{x})$:

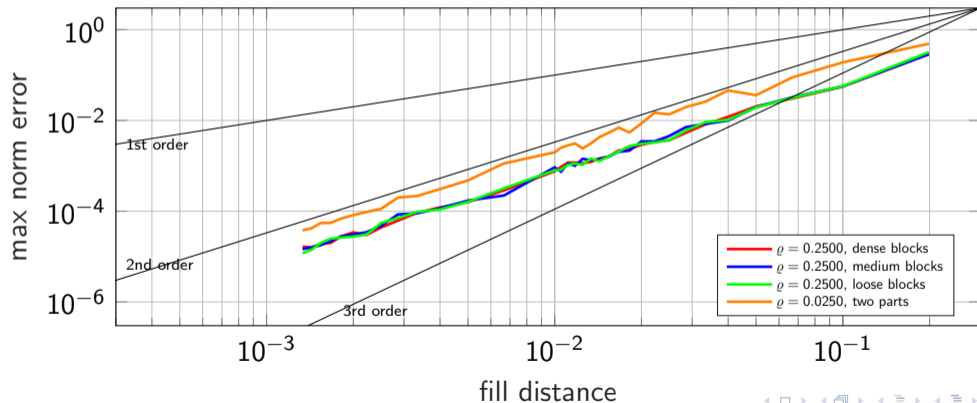
#	discription	λ/Λ for $\mathbf{x} \in [-1, 1]^2$
6	dense blocks	0.2500
7	medium blocks	0.2500
8	loose blocks	0.2500

We also tested the combination of previous matrices:

$$A_9(\mathbf{x}) = \begin{cases} A_2(\mathbf{x}), & x_1 < 0 \\ A_3(\mathbf{x}), & \text{otherwise} \end{cases} \quad \text{with} \quad \begin{cases} \lambda/\Lambda = 0.0250 \\ \text{for } \mathbf{x} \in [-1, 1]^2 \end{cases}$$

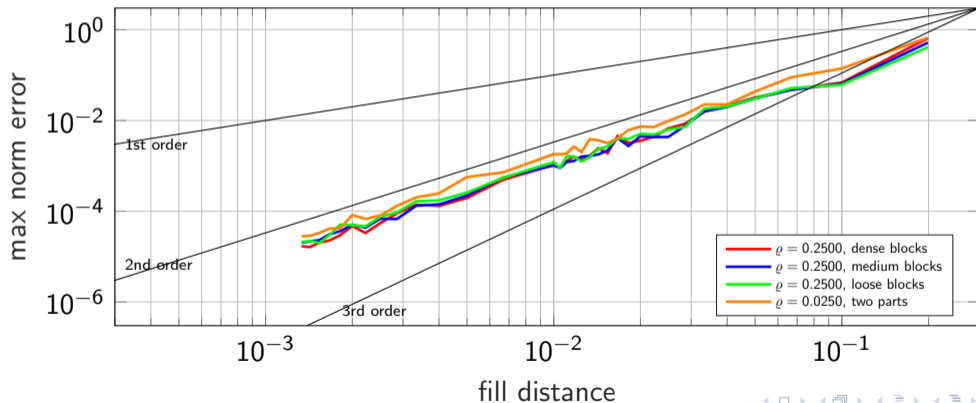
Error Graph for Discontinuous Coefficient Matrices

Unit Disk Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$

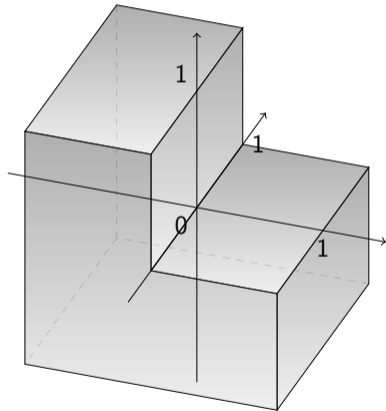
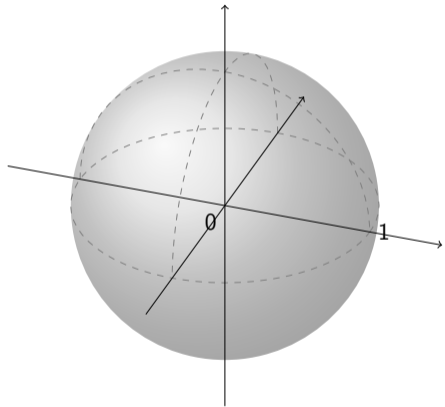


Error Graph for Discontinuous Coefficient Matrices

L-shaped Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$

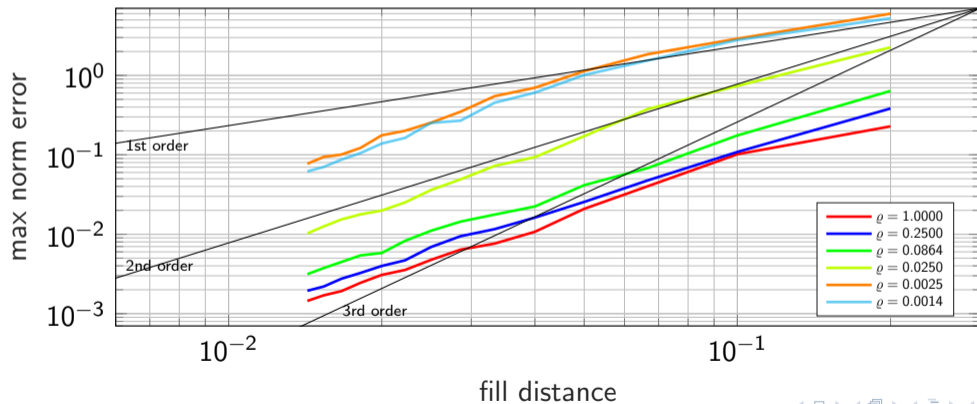


Domains



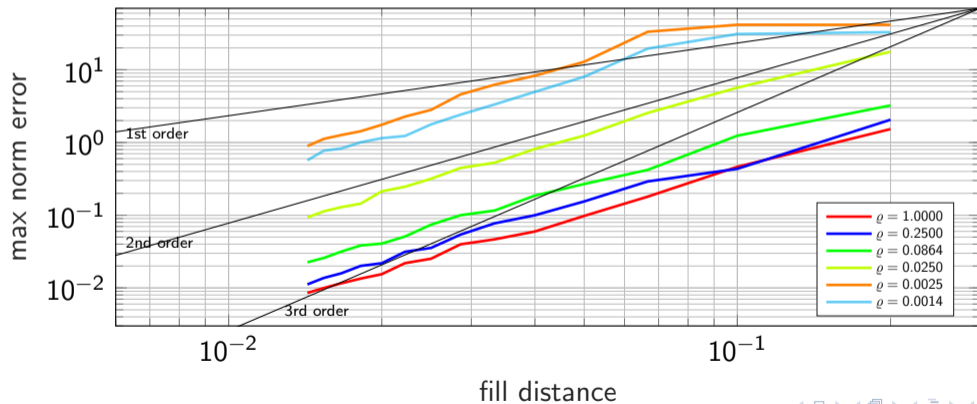
Error Graph for Continuous Coefficient Matrices

Unit Sphere Domain, $d = 3, p = 2, u(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 \cos(x_1(x_1 + 2x_2 + 2x_3))$



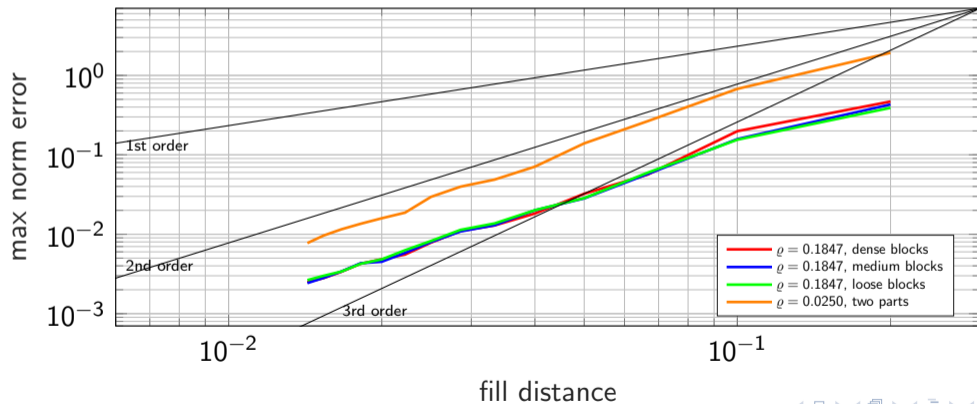
Error Graph for Continuous Coefficient Matrices

3d L-shaped Domain, $d = 3, p = 2, u(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 \cos(x_1(x_1 + 2x_2 + 2x_3))$



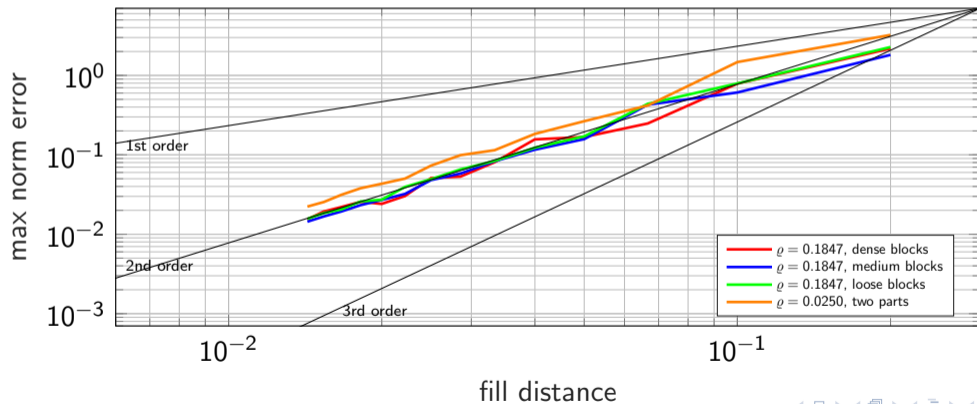
Error Graph for Discontinuous Coefficient Matrices

Unit Sphere Domain, $d = 3, p = 2, u(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 \cos(x_1(x_1 + 2x_2 + 2x_3))$



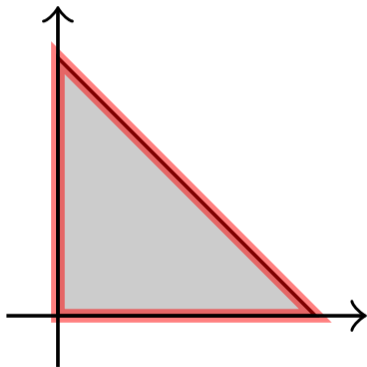
Error Graph for Discontinuous Coefficient Matrices

3d L-shaped Domain, $d = 3, p = 2, u(x_1, x_2, x_3) = (x_1 + x_2 + x_3)^4 \cos(x_1(x_1 + 2x_2 + 2x_3))$



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Kimura Diffusion Operator¹¹

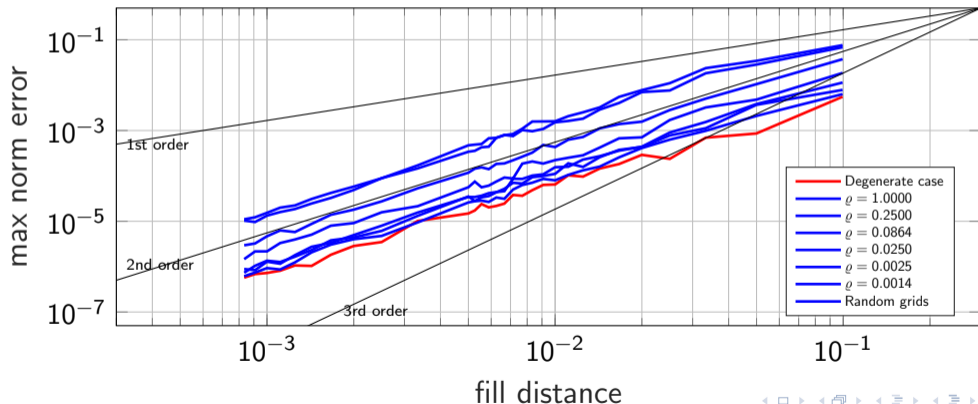
$$L_{\text{Kim}} = \sum_{i,j=1}^d x_i(\delta_{ij} - x_j)\partial_{x_i}\partial_{x_j}$$

$$A(\mathbf{x}) = \begin{pmatrix} x_1(1-x_1) & -x_1x_2 & \cdots & -x_1x_d \\ -x_2x_1 & x_2(1-x_2) & \cdots & -x_2x_d \\ \vdots & \vdots & \ddots & \vdots \\ -x_dx_1 & -x_dx_2 & \cdots & x_d(1-x_d) \end{pmatrix}$$

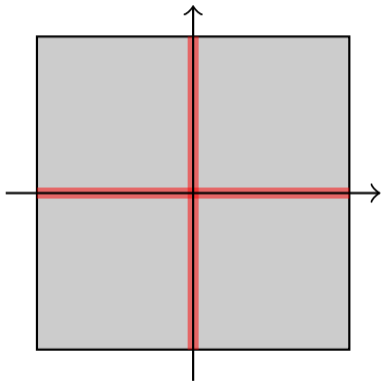
¹¹[Epstein and Mazzeo, 2013]

Error Graph for Kimura Diffusion Operator

Simplex Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$



An Example of Internal Degeneration¹²

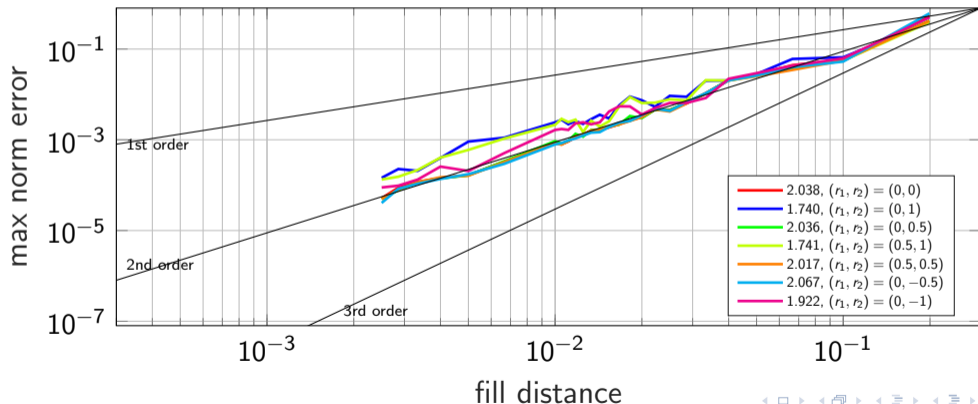


$$A(\mathbf{x}) = \begin{pmatrix} |x_1|^{r_1} & 0 & \cdots & 0 \\ 0 & |x_2|^{r_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & |x_d|^{r_d} \end{pmatrix}$$

¹²[Nazarov, 2010]

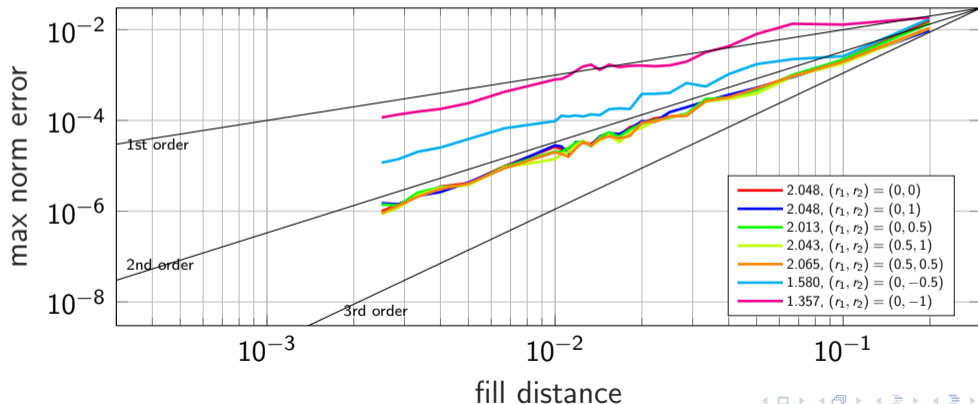
Error Graph for a Smooth Solution

Square Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 + x_2)^4 \cos(x_1(x_1 + 2x_2))$



Error Graph for a Non-smooth Solution

Square Domain, $d = 2, p = 2, u(x_1, x_2) = (x_1 x_2)^{2.5}$



Outline

- 1 Introduction
 - Background
- 2 Basic Ideas
 - Meshfree Finite Difference Method
 - Analytical Results
- 3 Numerical Results
 - 2d Examples
 - 3d Examples
- 4 Degenerate Case Examples
 - Boundary Degeneration
 - Internal Degeneration
- 5 Conclusion

Main Results

- 1 We have developed a meshfree method for solving non-divergence elliptic PDEs via nonlocal relaxation.
- 2 Minimal positive stencils are created to guarantee the numerical stability and efficiency of the method.
- 3 Theoretical convergence is established and second order (super)convergence is observed for $p = 2$ in practice.
- 4 The method can work for discontinuous coefficient matrices, non-convex domains, and (near) degenerate elliptic equations.
- 5 Our study improves the known theoretical results on the existence of positive stencils for linear elliptic equations when the ellipticity constant becomes small.

Future Work

- 1 Theoretical analysis for the super-convergence phenomenon.
- 2 High order methods; adaptive refinements; other boundary conditions.
- 3 Convergence analysis for degenerate elliptic equations.
- 4 Extensions to elliptic PDEs on manifolds and nonlinear elliptic PDEs.

Thank you



Scan to see interactive examples and more numerical results

Or visit <https://yee172.com/MPS4PDEs>

Manuscript:

<http://doi.org/10.1007/s10915-023-02294-3>

Thank you for listening!

Questions?

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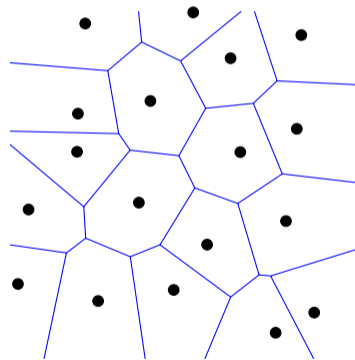
Proper Point Cloud

Fill Distance:

$$h = \inf \left\{ h : \overline{\Omega_\delta} \subseteq \bigcup_{i=1}^M \overline{B_h(\mathbf{x}_i)} \right\}$$

$$= \sup_{\mathbf{x} \in \overline{\Omega_\delta}} \min_{1 \leq i \leq M} |\mathbf{x} - \mathbf{x}_i|.$$

Use **Voronoi diagram**.



Proper Point Cloud (cont.)

When we say a point cloud is proper, we mean

- 1 Fill distance is small enough;
- 2 Separation between interior points is large enough;
- 3 Proportional to the fill distance, there are no interior points too close to the boundary.

In short, we want the points in the point cloud to be as evenly distributed as possible (each point is not too far away from or too close to its neighbors).

Cone Condition

Theorem 3 (Seibold, 2008)

Take a point cloud $X = \{\mathbf{x}_i\}_{i=1}^M \subset \Omega_\delta \subset \mathbb{R}^d$ and let $\mathbf{x}_i \in \Omega$ be fixed. If for any unit vector $\mathbf{v} \in \mathbb{R}^d$, $\mathcal{C}_\delta^\mathbf{v}(\mathbf{x}_i) \cap X \setminus \{\mathbf{x}_i\} \neq \emptyset$, then the feasible set with $A(\mathbf{x}) = I_{d \times d}$ and $p = 2$ is not empty.

The cone $\mathcal{C}_\delta^\mathbf{v}(\mathbf{x}_i)$ in $B_\delta(\mathbf{x}_i)$ is defined as

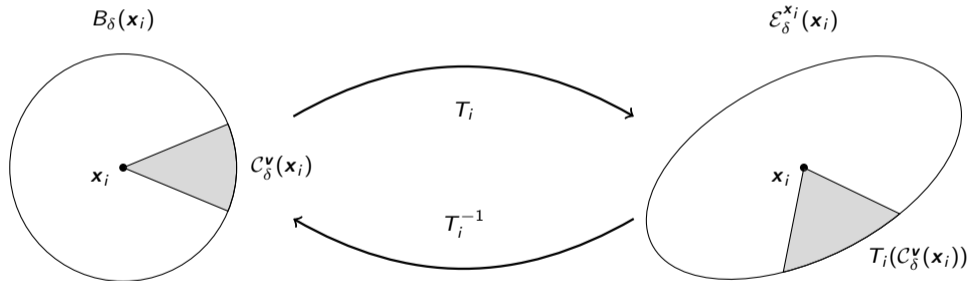
$$\mathcal{C}_\delta^\mathbf{v}(\mathbf{x}_i) := \left\{ \mathbf{x} \in B_\delta(\mathbf{x}_i) : \mathbf{x}^T \mathbf{v} \geq \frac{1}{\sqrt{1 + \sigma_d}} |\mathbf{x}|^2 \right\}$$

where $\sigma_d = \sqrt{2} - 1$ (a cone with total opening angle 45°) for $d = 2$ and

$\sigma_d = \sqrt{(3 - \sqrt{6})/6}$ (a cone with total opening angle 33.7°) for $d = 3$.

Cone Condition (cont.)

Using mapping, we can extend the cone condition to the general case.



Cone Condition (cont.)

By the definition of the fill distance, there are no holes with a radius larger than h . We can estimate how large the searching area is needed. We can prove that $r(\varrho) = \Theta(\sqrt{\varrho})$.

